Physics Unlimited Premier Competition
2021 Examination Solutions

November 14, 2021

This solution document contains 16 pages (including this cover page) and 5 questions worth 85 points total.

For questions with many sub-questions, please be lenient when grading. If student makes an algebraic mistake and it propagates to the next parts, only deduct points for the first instance of the mistake, unless it alters the solution significantly.

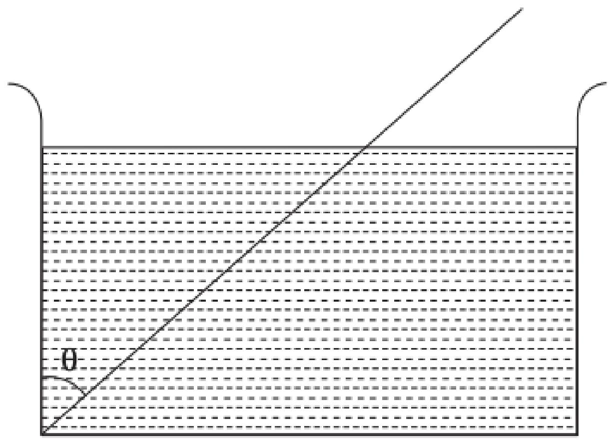
Partial credit can be awarded. Physical intuition and good argumentation are worthy of consideration even in the absence of sufficient mathematical support. This can especially be relevant for problem 5. Do it when you see fit, but try to be consistent as it can be challenging in some cases.

Distribution of Marks

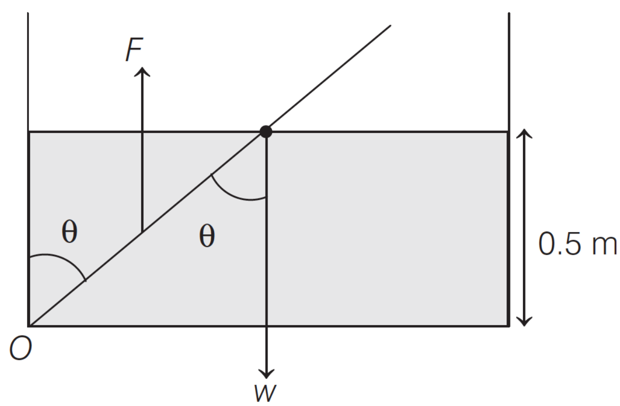
Question	Points	Score
1	10	
2	15	
3	15	
4	20	
5	25	
Total:	85	

1. (10 points) **Restore equilibrium (10 points)**

A wooden plank of length 1 m and uniform cross-section is hinged at one end to the bottom of a tank as shown in figure. The tank is filled with water up to a height 0.5 m. The specific gravity of the plank (or ratio of plank density to water density) is 0.5. **Find the angle θ that the plank makes with the vertical in the equilibrium position** (exclude the case $\theta = 0°$)



Solution:



One could find the net buoyant force by integrating about the point of reference. Or one can directly use the fact this upthrust force acts at the center of portion submerged in water.

A sample solution:

Three forces will act on the plank-

1. Weight which will act at centre of plank.

2. Upthrust which will act at centre of submerged portion.
3. Force from the hinge at O.

Notation used: Submerged length = 0.5 sec θ, F = Upthrust, w = Weight

Taking moments of all three forces about point O. Moment of hinge force will be zero.

$$(Alg\rho)\frac{l}{2}sin\theta = A(0.5sec\theta)(\rho_w)(g)\left(\frac{0.5sec\theta}{2}\right)sin\theta \tag{1}$$

Solve to get $\theta = 45°$

2. **State secrets and the stars (15 points)**

In 1949, the physicist G. I. Taylor used publicly released images of the Trinity nuclear test to estimate the yield of the world's first atomic bomb (i.e., the energy released by the detonation). His estimate was stunningly accurate, despite the fact that the number itself was still classified by the government at the time. *In this problem, we will follow a simplified version of his estimate, and use the procedure to estimate the energy released by the supernovae which end the lives of massive stars.*

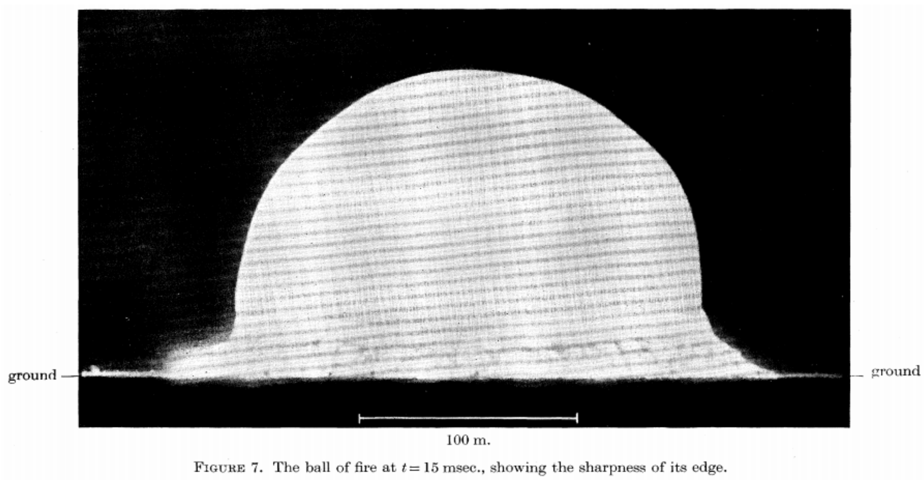


Figure 1: A snapshot of the Trinity blast wave 15 milliseconds after the explosion, reproduced from Taylor's original paper. A helpful scale bar is included.

(a) (5 points) After a very short moment, the shock front caused by an explosion sweeps up the material around it, increasing the mass which is blasted outwards. At early times, the energy $E \sim Mv^2/2$ can be thought of as roughly conserved.

Let R be the radius of the explosion at some time t. Estimate the speed of the shock front as $v \sim R/t$ and the swept up mass $M \sim \rho R^3$, where ρ is the density of the surrounding material (don't bother to include factors like $4\pi/3$).

Estimate the radius of the explosion R(t) as a function of the energy yield E, elapsed time t, and the density of the surrounding medium ρ.

(b) (3 points) Air has a density $\rho \simeq 1\,\mathrm{kg\,m^{-3}}$. Using the provided image, **roughly estimate the yield of the Trinity explosion in kilotons (of TNT).**

Note that 1 kiloton is equal to approximately 4.2×10^{12} J.

(c) (7 points) In 1054, Chinese astronomers observed and documented a supernova which was bright enough to be visible during the day for around a month. The rubble left behind is a rapidly expanding supernova remnant called the **Crab Nebula**, which is intensely studied today.

The surrounding interstellar medium contains about 1 proton per cubic centimeter—protons have a mass $m_p \approx 1.7 \times 10^{-27}\,\mathrm{kg}$. While not perfectly spherical, Crab Nebula has a radius of roughly $1.7\,\mathrm{pc}$ ($1\,\mathrm{pc} = 3.1 \times 10^{16}\,\mathrm{m}$). From this information and the given

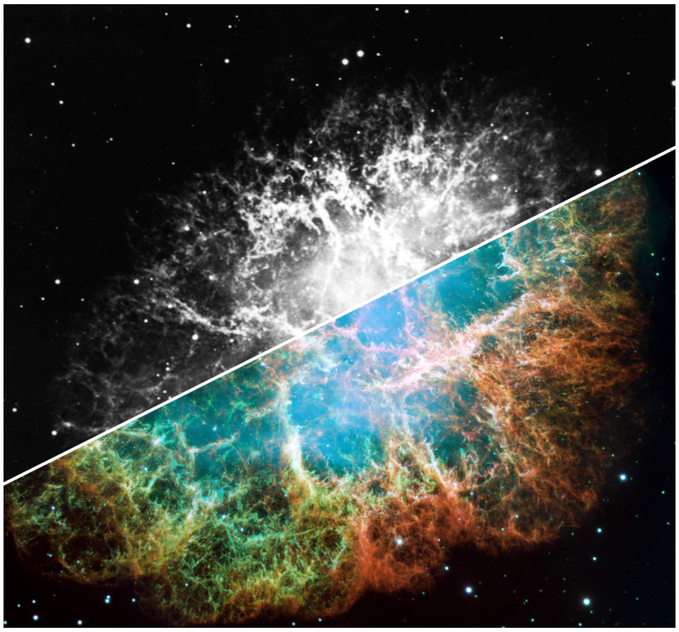


Figure 2: Photographs of the Crab Nebula taken by the Hale Telescope in 1950 (top left) and the Hubble Space Telescope in 2000 (bottom right).

images of the Crab Nebula from 1950 and 2000, **estimate the energy released by the initial supernova in joules.**

Your answer should be close to the typical energy scales of supernovae, which briefly outshine their host galaxies and can be seen from across the universe.

Solutions:

(a) Full points should be awarded to a solution as below up to order unity factors.
The energy of the explosion is $E = Mv^2/2 \sim Mv^2$. Using $v \sim R/t$ and $M \sim \rho R^3$, the total energy of the explosion is

$$E \sim \frac{\rho R^5}{t^2} \tag{2}$$

We can then rearrange this expression as

$$R(t) \sim \left(\frac{E}{\rho}\right)^{1/5} t^{2/5} \tag{3}$$

(b) From the included figure, the explosion looks to be approximately ~ 100 m in radius at a time ~ 15 ms. We can use Equation 2 with the given values to obtain

$$E \sim 11\,\text{kilotons of TNT} \tag{4}$$

Note that this value may vary significantly depending on the estimated radius from the figure (since $E \propto R^5$ depends strongly on R), but should be in the neighborhood of tens of kilotons of TNT. The official number released by the government at the time was 21 kilotons of TNT, although there has been some uncertainty in this number.

(c) The density of the interstellar medium is

$$\rho \sim \frac{m_p}{(1\,\text{cm}^3)} \sim 1.7 \times 10^{-21}\,\text{kg}\,\text{cm}^{-3} \tag{5}$$

We are given that the Crab Nebula has a radius of approximately 1.7 pc. We see that the explosion seems to expand by $\sim 25\%$ or so during the fifty years of observation along the long axis, which we take to be a very rough measure of its velocity. In particular,

$$v \sim \frac{25\% \times 1.7\,\text{pc}}{50\,\text{yr}} \sim 8.3 \times 10^6\,\text{m}\,\text{s}^{-1} \tag{6}$$

We can use $E \sim Mv^2$ and $M \sim \rho R^3$ to write

$$E \sim \rho R^3 v^2 \sim 1.7 \times 10^{43}\,\text{J} \tag{7}$$

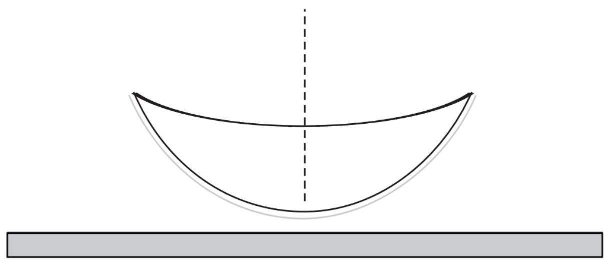
where we have used $t \sim 1000$ yr (the age of the supernova remnant). This is $E \sim 1.7 \times 10^{50}\,\text{erg}$, which is in the neighborhood of true supernovae which are typically at $\sim 10^{51}\,\text{erg}$ (a quantity of energy which has been affectionately named the "f.o.e." by astrophysicists because of this fact).

3. **Stick a pin there (15 points)**

The convex surface of a thin concavo-convex lens of glass of refractive index 1.5 has a radius of curvature 20 cm. The concave surface has a radius of curvature 60 cm. The convex side is silvered and placed on a horizontal surface.

(a) (7 points) Where should a pin be placed on the optical axis such that its image is formed at the same place?

(b) (8 points) If the concave part is filled with water of refractive index 4/3, find the distance through which the pin should be moved, so that the image of the pin again coincides with the pin.



Solutions:

(a) Image of object will coincide with it if ray of light after refraction from the concave surface fall normally on concave mirror so formed by silvering the convex surface. Or image after refraction from concave surface should form at centre of curvature of concave mirror or at a distance of 20 cm on same side of the combination. Let x be the distance of pin from the given optical system.

Using,

$$\frac{\mu_2}{\nu} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R} \tag{8}$$

With proper signs,

$$\frac{1.5}{-20} - \frac{1}{-x} = \frac{1.5 - 1}{-60} \tag{9}$$

Solve to get $x = 15$ cm

(b) There could be different approaches to solve this. A sample solution:

Now, before striking with the concave surface, the ray is first refracted from a plane surface. So, let x be the distance of pin, then the plane surface will form its image at a distance $\frac{4}{3}x(h_{app} = \mu h)$ from it.
Using,

$$\frac{\mu_2}{\nu} - \frac{\mu_1}{u} = \frac{\mu_2 - \mu_1}{R} \tag{10}$$

with proper signs,

$$\frac{1.5}{-20} - \frac{4/3}{-4x/3} = \frac{1.5 - 4/3}{-60} \tag{11}$$

Solve to get $x = 13.84$ cm.
Therefore $\Delta x = x_1 - x_2 = 15\ cm - 13.84\ cm = 1.16\ cm$ **(Downwards)**

4. **A complex dance (20 points)**

In this problem, we will solve a number of differential equations corresponding to very different physical phenomena that are unified by the idea of oscillation. Oscillations are captured elegantly by extending our notion of numbers to include the **imaginary** unit number i, strangely defined to obey $i^2 = -1$. In other words, rather than using **real** numbers, it is more convenient for us to work in terms of **complex** numbers.

Exponentials are usually associated with rapid growth or decay. However, with the inclusion of complex numbers, imaginary "growth" and "decay" can be translated into **oscillations** by the **Euler identity**:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{12}$$

(a) (3 points) The usual form of Newton's second law ($\vec{F} = m\vec{a}$) breaks down when we go into a rotating frame, where both the centrifugal and Coriolis forces become important to account for. Newton's second law then takes the form

$$\vec{F} = m\left(\vec{a} + 2\vec{v}\times\vec{\Omega} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})\right) \tag{13}$$

For a particle free of forces confined to the x–y plane in a frame which rotates about the z axis with angular frequency Ω, this becomes the complicated-looking system of differential equations,

$$\begin{aligned} 0 &= \ddot{x} + 2\Omega\dot{y} - \Omega^2 x \\ 0 &= \ddot{y} - 2\Omega\dot{x} - \Omega^2 y \end{aligned} \tag{14}$$

where dots represent time derivatives.

Defining $\eta = x + iy$, **show that Equations 14 are equivalent to the following single (complex) equation:**

$$0 = \ddot{\eta} - 2i\Omega\dot{\eta} - \Omega^2\eta \tag{15}$$

(b) (3 points) Equation 15 is a version of the **damped harmonic oscillator**, and can be solved by guessing a solution $\eta = \alpha e^{\lambda t}$.

Plugging in this guess, what must λ be?

(c) (4 points) Using your answer to part (b), and defining $\alpha = Ae^{i\phi}$ where A and ϕ are real, **find $\mathbf{x(t)}$ and $\mathbf{y(t)}$.**

This is the trajectory for a particle which is stationary with respect to the symmetry axis. While not required for this problem, an additional guess would reveal that $\eta = \beta t e^{\lambda t}$ is also a solution.

(d) (3 points) The one-dimensional **diffusion equation** (also called the "heat equation") is given (for a free particle) by

$$\frac{\partial\psi}{\partial t} = a\frac{\partial^2\psi}{\partial x^2} \tag{16}$$

A spatial wave can be written as $\sim e^{ikx}$ (larger k's correspond to waves oscillating on smaller length scales). Guessing a solution $\psi(x,t) = Ae^{ikx - i\omega t}$, **find ω in terms of k.** A relationship of this time is called a "dispersion relation."

(e) (2 points) The most important equation of non-relativistic quantum mechanics is the **Schrödinger equation**, which is given by

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \tag{17}$$

Using your answer to part (d), **what is the dispersion relation of the Schrödinger equation?**

(f) (2 points) If the energy of a wave is $E = \hbar\omega$ and the momentum is $p = \hbar k$, **show that the dispersion relation found in part (e) resembles the classical expectation for the kinetic energy of a particle, $\mathbf{E = mv^2/2}$.**

(g) (3 points) The theory of relativity instead posits that the energy of a particle is given by $E = \sqrt{p^2c^2 + m^2c^4}$. In accordance with this, we can try to guess a relativistic version of the Schrödinger equation:

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \frac{m^2c^2}{\hbar^2}\phi = 0 \tag{18}$$

This is called the **Klein–Gordon equation.** Using the same guess as before, **find ω in terms of k.**

Hint: *If you are careful, you should find that there is an infinite continuum of energy states extending down to negative infinity. This apparently mathematical issue hints at the existence of antimatter, and ultimately demonstrates to us that we must formulate* ***quantum field theory*** *to properly describe relativistic quantum physics.*

Solutions:

(a) We can write down the equations of motion, multiplying the second one by i:

$$\begin{aligned} 0 &= \ddot{x} + 2\Omega\dot{y} - \Omega^2 x \\ 0 &= \ddot{y} - 2\Omega i\dot{x} - \Omega^2 iy \end{aligned} \tag{19}$$

We can add these equations together to obtain

$$0 = (\ddot{x} + i\ddot{y}) + 2\Omega(\dot{y} - i\dot{x}) - \Omega^2(x + iy) = \ddot{\eta} + 2\Omega(\dot{y} - i\dot{x}) - \Omega^2\eta \tag{20}$$

We note that

$$\dot{y} - i\dot{x} = -i(\dot{x} + i\dot{y}) = -i\eta \tag{21}$$

Then

$$0 = \ddot{\eta} - 2i\Omega\dot{\eta} - \Omega^2\eta \tag{22}$$

(b) We can plug in $\eta = \alpha e^{\lambda t}$:

$$0 = \lambda^2\alpha e^{\lambda t} - 2i\lambda\Omega\alpha e^{\lambda t} - \Omega^2\alpha e^{\lambda t} \tag{23}$$

Then we can cancel common factors and find that

$$0 = \lambda^2 - 2i\lambda\Omega - \Omega^2 = (\lambda - i\Omega)^2 \tag{24}$$

We see that $\lambda = i\Omega$.

(c) Using our answer to part (b) and $\alpha = Ae^{i\phi}$, we have

$$\eta(t) = Ae^{i(\Omega t + \phi)} \tag{25}$$

Using the Euler identity, we have

$$x(t) + iy(t) = A\cos(\Omega t + \phi) + iA\sin(\Omega t + \phi) \tag{26}$$

The real and imaginary parts become

$$\begin{aligned} x(t) &= A\cos(\Omega t + \phi) \\ y(t) &= A\sin(\Omega t + \phi) \end{aligned} \tag{27}$$

(d) Consider the given differential equation:

$$\frac{\partial\psi}{\partial t} = a\frac{\partial^2\psi}{\partial x^2} \tag{28}$$

We can plug in $\psi(x,t) = Ae^{ikx - i\omega t}$ to find

$$-i\omega Ae^{ikx - i\omega t} = -k^2 aAe^{ikx - i\omega t} \tag{29}$$

so that

$$\omega = -ik^2 a \tag{30}$$

(e) We see that the free Schrödinger equation takes the form of Equation 28, but with

$$a = \frac{i\hbar}{2m} \tag{31}$$

Then, using our answer from part (d), we have

$$\omega = \frac{\hbar k^2}{2m} \tag{32}$$

(f) We can multiply both sides of the answer to part (e) by $\hbar$ and use $E = \hbar\omega$ and $p = \hbar k$, we have

$$E = \frac{p^2}{2m} \tag{33}$$

A classical momentum has $p = mv$, so this gives the classical energy for a free particle (i.e., one without a potential):

$$E = \frac{1}{2}mv^2 \tag{34}$$

(g) We consider the Klein–Gordon equation:

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \frac{m^2c^2}{\hbar^2}\phi = 0 \tag{35}$$

We can, as before, plug in a guess $\phi(x,t) = Ae^{ikx-i\omega t}$. This yields

$$-\frac{\omega^2}{c^2}Ae^{ikx-i\omega t} + k^2Ae^{ikx-i\omega t} + \frac{m^2c^2}{\hbar^2}Ae^{ikx-i\omega t} = 0 \tag{36}$$

Cancelling out common terms, we see that

$$\omega^2 = k^2c^2 + \frac{m^2c^4}{\hbar^2} \tag{37}$$

or

$$\omega = \pm\sqrt{k^2c^2 + \frac{m^2c^4}{\hbar^2}} \tag{38}$$

Note that both positive and negative ω solve the Klein–Gordon equation.

5. **Polarization and Oscillation (25 points)**

In this problem, we will understand the polarization of metallic bodies and the method of images that simplifies the math in certain geometrical configurations.

Throughout the problem, suppose that metals are excellent conductors and they polarize significantly faster than the classical relaxation of the system.

(a) (1 point) Explain in words why **there can't be a non-zero electric field** in a metallic body, and why this leads to constant electric potential throughout the body.

(b) (2 points) Laplace's equation is a second order differential equation

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{39}$$

Solutions to this equation are called harmonic functions. One of the most important properties satisfied by these functions is **the maximum principle**. It states that a harmonic function attains **extremes on the boundary**.

Using this, **prove the uniqueness theorem**: Solution to Laplace's equation in a volume V is uniquely determined if its solution on the boundary is specified. That is, if $\nabla^2 \phi_1 = 0$, $\nabla^2 \phi_2 = 0$ and $\phi_1 = \phi_2$ on the boundary of V, then $\phi_1 = \phi_2$ in V.

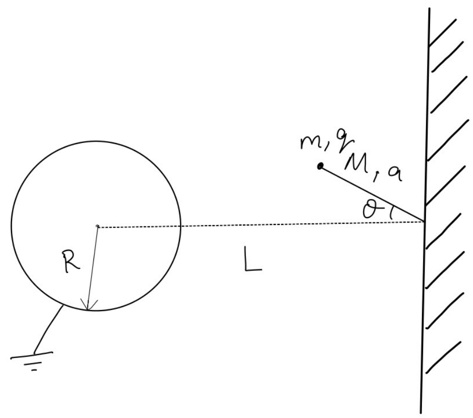
Hint: *Consider* $\phi = \phi_1 - \phi_2$.

(c) (4 points) The uniqueness theorem allows us to use "image" charges in certain settings to describe the system. Consider one such example: There is a point-like charge q at a distance L from a metallic sphere of radius R attached to the ground. As you argued in part (a), sphere will be polarized to make sure the electric potential is constant throughout its body. Since it is attached to the ground, the constant potential will be zero. Place an image charge inside the sphere to counter the non-uniform potential of the outer charge q on the surface. **Where should this charge be placed, and what is its value?**

(d) (1 point) Argue from the uniqueness theorem that the electric field created by this image charge outside the sphere will be the **same** as the field created by the complicated polarization of the sphere.

(e) (3 points) **Find the force** of attraction between the charge and the sphere.

(f) (14 points) Now suppose that we attach the point-like charge to a wall with a rod of length a. Any perturbation from the equilibrium will cause a perturbation of the polarization of the sphere. **Prove that this equilibrium is stable and find the frequency of oscillation around it.** The charge and rod have masses m and M, respectively and assume that $L > R + a$.



Solutions:

(a) In conductors, outer electron of each atom are free to move around the body. So if there is a non-zero electric field inside, this would move those free charges to the surface, at which point they cannot escape further, and they would accumulate there. This process would stop when the electric field created by the polarized charges exactly counter the external electric field inside. Consequently, any external electric field would cause a polarization of the body that results in a net zero field inside. Since $\nabla\phi = -\mathbf{E} = \mathbf{0}$, we have constant ϕ inside. From the continuity, it is also constant and equal to ϕ_{inside} throughout the surface.

(b) Suppose that we have two potentials satisfying the Laplace's equation and the boundary conditions. $\nabla^2\phi_1 = 0$, $\nabla^2\phi_2 = 0$ and $\phi_1 = \phi_2$ on the boundary of V. Define $\phi_3 = \phi_1 - \phi_2$. By the linearity of Laplace's equation, $\nabla^2\phi_2 = 0$ and $\phi_3 = 0$ on the surface. By the maximum principle, all extrema occurs at the surface. Thus, both the minimum and the maximum of ϕ_3 is 0. Therefore, $\phi_3 = 0$. This concludes the proof that $\phi_1 = \phi_2$ identically.

(c) Take the origin of our coordinate system at the center of the sphere and align the axes so that the coordinates of our charge is $(x, y) = (L, 0)$. Now suppose we place an image charge q_0 at $(x, 0)$ where $0 < r < R$. Then the potential at the point $(Rcos\theta, Rsin\theta)$ on the surface becomes

$$V(\theta) = \frac{q_0}{x_\theta} + \frac{q}{L_\theta}$$

where

$$x_\theta = \sqrt{x^2 + R^2 - 2xR\cos\theta}$$
$$L_\theta = \sqrt{L^2 + R^2 - 2LR\cos\theta}$$

We want this image charge to satisfy $V(\theta) = 0\ \forall\ \theta \in [0, 2\pi]$. Then we have,

$$\frac{q}{q_0} = -\frac{\sqrt{L^2 + R^2 - 2LR\cos\theta}}{\sqrt{x^2 + R^2 - 2xR\cos\theta}} = -\sqrt{\frac{L}{x}}\frac{\sqrt{\dfrac{L^2 + R^2}{L} - 2R\cos\theta}}{\sqrt{\dfrac{x^2 + R^2}{x} - 2R\cos\theta}}$$

Consequently,

$$\frac{L^2 + R^2}{L} = \frac{x^2 + R^2}{x} \quad \text{and}$$
$$\frac{q}{q_0} = -\sqrt{\frac{L}{x}}$$

Solution to the first equation with $x < R$ condition gives $x = \dfrac{R^2}{L}$, and from the second equation, q_0 can be found as $q_0 = -q\dfrac{R}{L}$. By azimuthal symmetry, we conclude that if the potential is zero on surface crossing the $x - y$ plane with the image charge $q_0 = -q\dfrac{R}{L}$ at $x = \dfrac{R^2}{L}$, then it is zero everywhere throughout the surface.

(d) As argued in part (a), polarization of the sphere would create an equipotential surface. If we consider the region outside the sphere, the image charge configuration satisfies the necessary boundary conditions: The inner boundary (surface of the sphere) has equipotential with $\phi = 0$, and the outer boundary (infinity) has zero potential. By the uniqueness theorem, potentials, and therefore electric fields, created by these different configurations are exactly the same throughout the volume in question.

(e) From part (d) and (c), this is simply

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2\dfrac{R}{L}}{\left(L - \dfrac{R^2}{L}\right)^2}$$

(f) Force is attractive. Therefore, any deviation from the equilibrium ($\theta = 0$) induces an attractive torque towards $\theta = 0$.
Force at the equilibrium would be the same as in part (e) with L replaced by $L - a$. Since we are looking for small oscillations, we need to calculate the torque in the first order approximation of θ. Denote β as the angle between the line segment from the center of the sphere to the particle and from the particle to the wall. Then the equations of motion become

$$\begin{aligned}\tau = -F(\theta)asin(\beta) &\approx -F(0)a\frac{L}{L-a}\theta \\ &= I\ \frac{d^2\theta}{dt^2} \\ &= \left(m + \frac{M}{3}\right)a^2\ \frac{d^2\theta}{dt^2}\end{aligned}$$

Thus it follows that

$$\omega = \frac{q}{(L-a)^2 - R^2}\sqrt{\frac{RL}{4\pi\epsilon_0 a\left(m + \dfrac{M}{3}\right)}}$$

In the limit $a << L$, we obtain the formula

$$\omega = \frac{q}{L^2 - R^2}\sqrt{\frac{RL}{4\pi\epsilon_0 a\left(m + \dfrac{M}{3}\right)}}$$