Physics Unlimited Premier Competition
2021 Examination

November 14, 2021

Fill in these two lines only if competing in-person:
Competitor ID (Exam Code): ____________________
Initials: ______________________________

This exam contains 9 pages (including this cover page) and 5 questions worth 85 points total.
Only start working on the exam when you are told to do so.
You are not expected to complete all of the exam. So you are encouraged to read through the exam first and start with the problems you find easiest. **The problems and their subparts are NOT ordered according to their difficulty, and it is possible that you can work out some later parts of a question when you are stuck on a former part. Don't spend too long on any one problem.** Partial credit will be awarded.

Please prepare at least 10 sheets of blank paper (for online competitors) or an empty exam booklet (provided to in-person competitors), a ruler for drawing graphs (in one of the problems), and a simple numeric calculator. Please be sure to not have anything else on your desk.

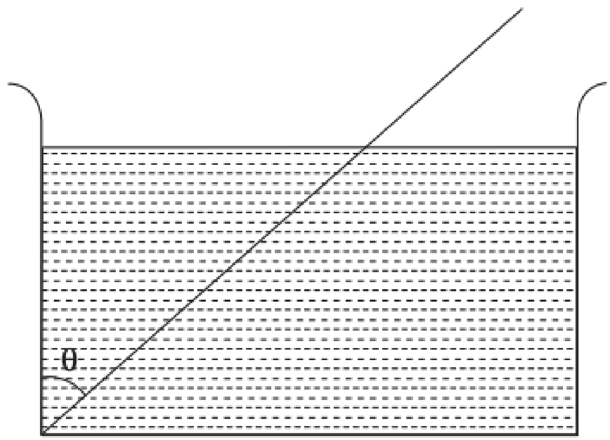
Note: if you are competing virtually, all work to be graded must be on blank sheets of paper that you will take photos of and submit as instructed immediately after the test. If you are competing in-person, all work should be in the exam workbook given to you. Box all answers, and try to work as clearly and neatly as possible.

Distribution of Marks

Question	Points	Score
1	10	
2	15	
3	15	
4	20	
5	25	
Total:	85	

1. (10 points) **Restore equilibrium (10 points)**

A wooden plank of length 1 m and uniform cross-section is hinged at one end to the bottom of a tank as shown in figure. The tank is filled with water up to a height 0.5 m. The specific gravity of the plank (or ratio of plank density to water density) is 0.5. **Find the angle θ that the plank makes with the vertical in the equilibrium position** (exclude the case $\theta = 0°$)



2. **State secrets and the stars (15 points)**

In 1949, the physicist G. I. Taylor used publicly released images of the Trinity nuclear test to estimate the yield of the world's first atomic bomb (i.e., the energy released by the detonation). His estimate was stunningly accurate, despite the fact that the number itself was still classified by the government at the time. *In this problem, we will follow a simplified version of his estimate, and use the procedure to estimate the energy released by the supernovae which end the lives of massive stars.*

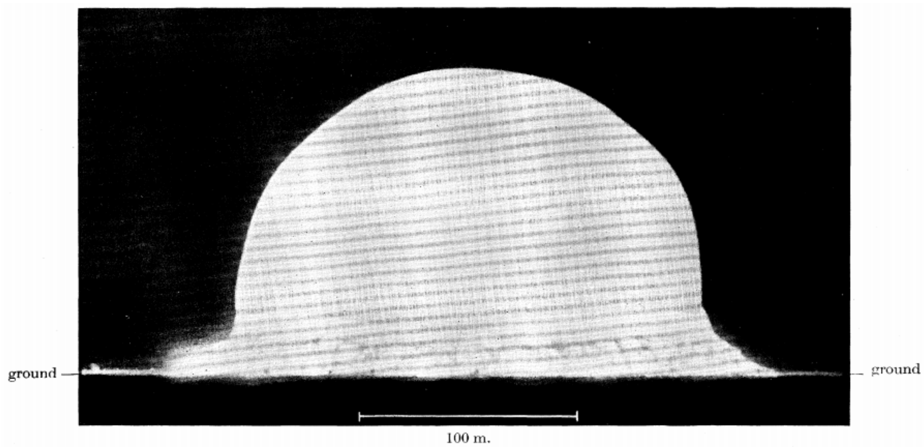


FIGURE 7. The ball of fire at $t = 15$ msec., showing the sharpness of its edge.

Figure 1: A snapshot of the Trinity blast wave 15 milliseconds after the explosion, reproduced from Taylor's original paper. A helpful scale bar is included.

(a) (5 points) After a very short moment, the shock front caused by an explosion sweeps up the material around it, increasing the mass which is blasted outwards. At early times, the energy $E \sim Mv^2/2$ can be thought of as roughly conserved.

Let R be the radius of the explosion at some time t. Estimate the speed of the shock front as $v \sim R/t$ and the swept up mass $M \sim \rho R^3$, where ρ is the density of the surrounding material (don't bother to include factors like $4\pi/3$).

Estimate the radius of the explosion R(t) as a function of the energy yield E, elapsed time t, and the density of the surrounding medium ρ.

(b) (3 points) Air has a density $\rho \simeq 1\,\mathrm{kg\,m^{-3}}$. Using the provided image, **roughly estimate the yield of the Trinity explosion in kilotons (of TNT).**

Note that 1 kiloton is equal to approximately 4.2×10^{12} J.

(c) (7 points) In 1054, Chinese astronomers observed and documented a supernova which was bright enough to be visible during the day for around a month. The rubble left behind is a rapidly expanding supernova remnant called the **Crab Nebula**, which is intensely studied today.

The surrounding interstellar medium contains about 1 proton per cubic meter—protons have a mass $m_p \approx 1.7 \times 10^{-27}$ kg. While not perfectly spherical, Crab Nebula has a radius of roughly 1.7 pc ($1\,\mathrm{pc} = 3.1 \times 10^{16}\,\mathrm{m}$). From this information and the given images of

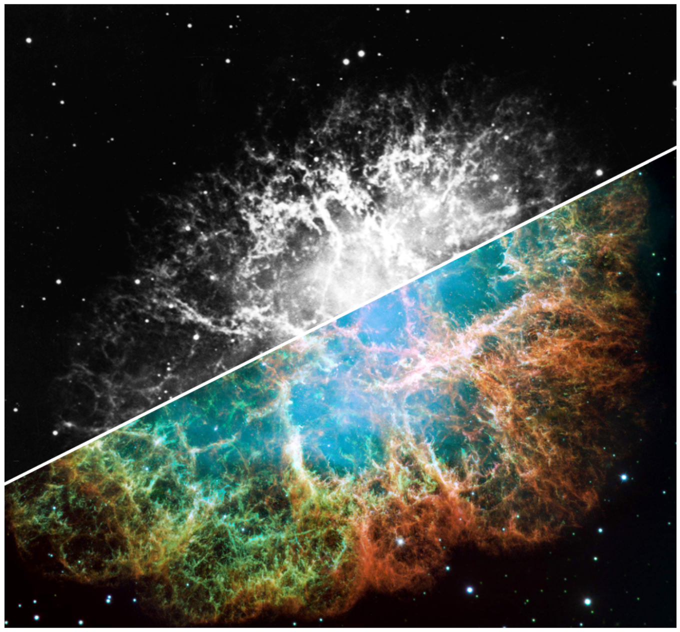


Figure 2: Photographs of the Crab Nebula taken by the Hale Telescope in 1950 (top left) and the Hubble Space Telescope in 2000 (bottom right).

the Crab Nebula from 1950 and 2000, **estimate the energy released by the initial supernova.**

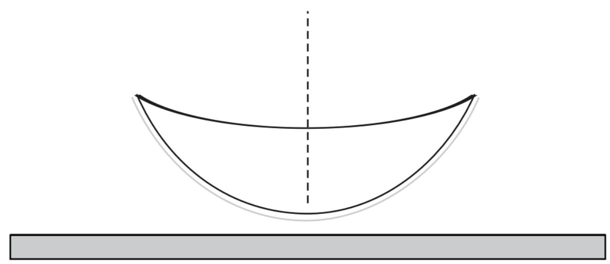
Your answer should be close to the typical energy scales of supernovae, which briefly outshine their host galaxies and can be seen from across the universe.

3. **Stick a pin there (15 points)**

The convex surface of a thin concavo-convex lens of glass of refractive index 1.5 has a radius of curvature 20 cm. The concave surface has a radius of curvature 60 cm. The convex side is silvered and placed on a horizontal surface.

(a) (7 points) Where should a pin be placed on the optical axis such that its image is formed at the same place?

(b) (8 points) If the concave part is filled with water of refractive index 4/3, find the distance through which the pin should be moved, so that the image of the pin again coincides with the pin.



4. **A complex dance (20 points)**

In this problem, we will solve a number of differential equations corresponding to very different physical phenomena that are unified by the idea of oscillation. Oscillations are captured elegantly by extending our notion of numbers to include the **imaginary** unit number i, strangely defined to obey $i^2 = -1$. In other words, rather than using **real** numbers, it is more convenient for us to work in terms of **complex** numbers.

Exponentials are usually associated with rapid growth or decay. However, with the inclusion of complex numbers, imaginary "growth" and "decay" can be translated into **oscillations** by the **Euler identity**:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1}$$

(a) (3 points) The usual form of Newton's second law ($\vec{F} = m\vec{a}$) breaks down when we go into a rotating frame, where both the centrifugal and Coriolis forces become important to account for. Newton's second law then takes the form

$$\vec{F} = m\left(\vec{a} + 2\vec{v}\times\vec{\Omega} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})\right) \tag{2}$$

For a particle free of forces confined to the x–y plane in a frame which rotates about the z axis with angular frequency Ω, this becomes the complicated-looking system of differential equations,

$$\begin{aligned} 0 &= \ddot{x} + 2\Omega\dot{y} - \Omega^2 x \\ 0 &= \ddot{x} - 2\Omega\dot{x} - \Omega^2 y \end{aligned} \tag{3}$$

where dots represent time derivatives.

Defining $\eta = x + iy$, **show that Equations 3 are equivalent to the following single (complex) equation:**

$$0 = \ddot{\eta} - 2i\Omega\dot{\eta} - \Omega^2\eta \tag{4}$$

(b) (3 points) Equation 4 is a version of the **damped harmonic oscillator**, and can be solved by guessing a solution $\eta = \alpha e^{\lambda t}$.

Plugging in this guess, what must λ be?

(c) (4 points) Using your answer to part (b), and defining $\alpha = Ae^{i\phi}$ where A and ϕ are real, **find $\mathbf{x(t)}$ and $\mathbf{y(t)}$.**

This is the trajectory for a particle which is stationary with respect to the symmetry axis. While not required for this problem, an additional guess would reveal that $\eta = \beta t e^{\lambda t}$ is also a solution.

(d) (3 points) The one-dimensional **diffusion equation** (also called the "heat equation") is given (for a free particle) by

$$\frac{\partial\psi}{\partial t} = a\frac{\partial^2\psi}{\partial x^2} \tag{5}$$

A spatial wave can be written as $\sim e^{ikx}$ (larger k's correspond to waves oscillating on smaller length scales). Guessing a solution $\psi(x,t) = Ae^{ikx - i\omega t}$, **find ω in terms of k.** A relationship of this time is called a "dispersion relation."

(e) (2 points) The most important equation of non-relativistic quantum mechanics is the **Schrödinger equation**, which is given by

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \tag{6}$$

Using your answer to part (d), **what is the dispersion relation of the Schrödinger equation?**

(f) (2 points) If the energy of a wave is $E = \hbar\omega$ and the momentum is $p = \hbar k$, **show that the dispersion relation found in part (e) resembles the classical expectation for the kinetic energy of a particle, $\mathbf{E = mv^2/2}$.**

(g) (3 points) The theory of relativity instead posits that the energy of a particle is given by $E = \sqrt{p^2c^2 + m^2c^4}$. In accordance with this, we can try to guess a relativistic version of the Schrödinger equation:

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \frac{m^2c^2}{\hbar^2}\phi = 0 \tag{7}$$

This is called the **Klein–Gordon equation.** Using the same guess as before, **find ω in terms of k.**

Hint: *If you are careful, you should find that there is an infinite continuum of energy states extending down to negative infinity. This apparently mathematical issue hints at the existence of antimatter, and ultimately demonstrates to us that we must formulate* ***quantum field theory*** *to properly describe relativistic quantum physics.*

5. **Polarization and Oscillation (25 points)**

In this problem, we will understand the polarization of metallic bodies and the method of images that simplifies the math in certain geometrical configurations.

Throughout the problem, suppose that metals are excellent conductors and they polarize significantly faster than the classical relaxation of the system.

(a) (1 point) Explain in words why **there can't be a non-zero electric field** in a metallic body, and why this leads to constant electric potential throughout the body.

(b) (2 points) Laplace's equation is a second order differential equation

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{8}$$

Solutions to this equation are called harmonic functions. One of the most important properties satisfied by these functions is **the maximum principle**. It states that a harmonic function attains **extremes on the boundary**.

Using this, **prove the uniqueness theorem**: Solution to Laplace's equation in a volume V is uniquely determined if its solution on the boundary is specified. That is, if $\nabla^2\phi_1 = 0$, $\nabla^2\phi_2 = 0$ and $\phi_1 = \phi_2$ on the boundary of V, then $\phi_1 = \phi_2$ in V.

Hint: *Consider* $\phi = \phi_1 - \phi_2$.

(c) (4 points) The uniqueness theorem allows us to use "image" charges in certain settings to describe the system. Consider one such example: There is a point-like charge q at a distance L from a metallic sphere of radius R attached to the ground. As you argued in part (a), sphere will be polarized to make sure the electric potential is constant throughout its body. Since it is attached to the ground, the constant potential will be zero. Place an image charge inside the sphere to counter the non-uniform potential of the outer charge q on the surface. **Where should this charge be placed, and what is its value?**

(d) (1 point) Argue from the uniqueness theorem that the electric field created by this image charge outside the sphere will be the **same** as the field created by the complicated polarization of the sphere.

(e) (3 points) **Find the force** of attraction between the charge and the sphere.

(f) (14 points) Now suppose that we attach the point-like charge to a wall with a rod of length a. Any perturbation from the equilibrium will cause a perturbation of the polarization of the sphere. **Prove that this equilibrium is stable and find the frequency of oscillation around it.** The charge and rod have masses m and M, respectively and assume that $L > R + a$.